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Watch π Appear and Disappear Magically

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Abstract

As in a magic show, the mathematical superstar π, defined as the ratio of circumference to diameter of a circle, aside from appearing in usual places, often shows up in unexpected places; likewise, occasionally, it is a no-show when its presence is intuitively anticipated. We give some examples of such magical appearances and disappearances. We hope curious readers will enjoy the show and inquisitive minds will seek to understand the explanations.

Keywords: Regular polygon, hyper-spheres, Gaussian distribution, Stirling's approximation, the central limit theorem, center of gravity

Mathematics Subject Classification 2020: 00A05

Prelude

In a well-executed magic show, the spectators are pleasantly surprised when, with a sleight of the magician's hand or a waving of the wand an item vanishes right in front of their eyes. They are equally astonished and awe-struck when an item suddenly appears from thin air.

I am no magician. Unlike my wife, who thinks magicians have superhuman powers, I surmise they have well-developed skills to divert the spectators' attention while their prior preparations, sneaky interventions, or invisible accomplishes carry out the task in secret. Therefore, when my wife prods me to explain how the magician did the trick, I let her win the argument whenever I have no answer. Often, I do not want to know the answer for fear that such knowledge will rob me of the pleasure of watching the magic show.

I am a simple mathematician — sometimes doing mathematics, but more often reading or watching other people's mathematics. I have noticed in many mathematical expressions the presence of π. Typically there is a direct or understandable explanation of its appearance. Sometimes the reasoning is long and more intricate. Once in a while, I have been taken aback by its sudden appearance when there was no prior hint. On the flip side, there are some instances when I was fully anticipating the presence of π; but it vanished into oblivion. Toward these situations, I wish to draw the reader's attention.

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I will explain some of the unexpected appearances or disappearances of π to reinforce these two truisms: (1) Although highly talented and worthy of respect, no mathematician has superhuman capabilities. (2) Although a superstar, π cannot appear or disappear at whim; it is bound by mathematical rules. However, often I will skip the details and give references instead.

Relax and enjoy the show. Section 1 presents the usual definition of π, its natural extensions, and its properties. Section 2 gives the anticipated presence of π in many situations involving a circle or an n-sphere. Section 3 lists some unexpected appearances of π — unrelated to a circle. Section 4 gives some strange disappearances of π when we anticipate its presence. Put your thinking cap on if and when so desired.

1 The Meanings of π and Its Powers

We begin with the most familiar definition of π, which has been in existence for over four millennia, and document its role in measuring items in two and higher dimensions.

1.1 The constant ratio in any circle permeates civilization

See Figure 1. For a circle of any size, the ratio of the circumference to the diameter is a constant, sometimes referred to as Archimedes's constant, denoted by

$$\pi = \frac{\text{circumference}}{\text{diameter}} \tag{1}$$

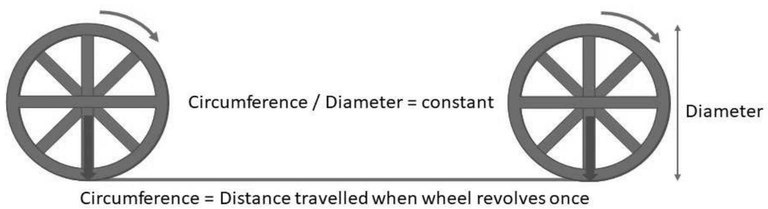

Figure 1: In any circle, the ratio of circumference to diameter is a constant.

At the dawn of the documented history of mathematics, about 2000 BCE, the existence and significance of this constant was known to (at least) Babylonians and Egyptians. See [3]. At different times and places, depending on differing needs of precision, Archimedes's constant has been variously approximated by $3, 22/7, 142/45, 333/106, 355/113, \sqrt{10}$, $3.14, 3.1416, \ldots$, etc. Below I give a mnemonic (each word length is the digit, with comma ignored and end-of-sentence punctuation read as 0) to help you remember the first 32 decimal places of

$\pi = 3.14159\,26535\,89793\,23846\,26433\,83279\,50\cdots$

"Yes, I find a sport broadcast so lively since our smart, prolific, energetic artiste practices the in-art joyfully when others do sourly over the air, assuming you do explore shortwave radio."

Readers will find some laudable piems (π-inspired poems) quoted in [2], including one by Wislawa Szymborska (1923–2012), the 1996 Nobel Laureate for Literature. Memorizing the digits of π is a popular challenge practiced on university campuses in the USA, especially around Pi Day (March 14), which, in 2009, the United States Congress declared as a national holiday.

The world record for π memorization, as certified by Guinness World Records, is held by Rajveer Meena, who recited 70,000 digits in 9 hours, 27 minutes on March 21, 2015. The unofficial world record (100,000 places) is held by Akira Haraguchi, who recited the digits from 2006 October 3, 9:00 a.m. to 2006 October 4, 1:28 a.m. (16 and 1/2 hours).

To track the evolution of knowledge, computation, and usage of π, see [1] and [36]. The quest for calculating π to many more digits is escalating. On Pi Day of 2019, tech giant Google announced that using their cloud platform π has been calculated to 31.4 trillion decimal places (or $10^{13}\,\pi$ places), only to be surpassed by Team DAVIS at the University of Applied Sciences of the Grisons, Switzerland, who doubled the number of digits on August 14, 2021. Within less than a year, on March 21, 2022, Emma Haruka Iwao extended the number of digits to 10^{14} places using a supercomputer at the University of Tokyo for 158 days. A year later, on 8 April 2023, the computations were verified in 59 days by Jordan Ranous.

The earliest known use of the Greek letter π to represent the constant was in 1706 by the self-taught Welsh mathematics teacher William Jones (1675–1749) in his second book [19]. It is the first letter of the Greek word "$\pi\epsilon\rho\iota\phi\epsilon\rho\epsilon\iota\alpha$" (peripheria), or the perimeter of a circle. Accordingly, today we write the circumference C as a function of the diameter d or the radius r of a circle as $C = \pi\, d = 2\pi\, r$. A curious two-part puzzle was floated in 1702:

> "A rope is wrapped around a desktop globe along its equator. How much longer does the rope have to be so that it will be one foot off the equator all the way around? What if the rope was initially wrapped around the earth's equator?"

In both cases, the answer is 2π feet, which is only about the height of a tall man and is independent of the size of the spherical object! The simple justification rests on the circumference formula: $C_1 = 2\pi\,(r+1) = 2\pi\, r + 2\pi = C_0 + 2\pi$.

Speaking of constant ratio in a circle, here is more: The circumference of a circle can be approximated by the perimeter of an inscribed regular n-gon (having n equal sides and n equal angles) and then by letting n increase *ad infinitum.* In a regular n-gon, of any size, the ratio of the perimeter to the length of the longest diagonal is a constant λ_n. Pawan Kumar [26] evaluates λ_n for $n = 3, 4, 5, \ldots, 5000$, and exhibits graphically that λ_n's converge to π.

1.2 The same constant appears in the area of a circle

Since ancient time, humans have known that the area of a circle is proportional to that of a square on its radius. Say, you wish to reduce your food intake. If you order an 8 inch pizza instead of your usual 10 inch pizza, you will eat 36% less than you usually eat.

Around (250 BCE) Archimedes (287–212/211 BCE) was the first to prove that the constant of proportionality is the same π defined in (1). See [5]. Here is how: If we join to the center the vertices of the regular n-gon inscribed in (or circumscribed about) a circle of radius r, then the n-gon is dissected into n triangles, whose total area approximates from below (or from above) the area of the circle. Passing to limit as $n \to \infty$, the perimeter p_n of a regular n-gon converges to the circumference $C = 2\pi r$ of the circle. Hence, the area of the circle (as an aggregate of the areas, half base times height, of n triangles) becomes

$$A = \frac{1}{2}\left(\lim_{n\to\infty} p_n\right)\, r = \frac{1}{2} C\, r = \pi\, r^2. \tag{2}$$

Equivalently, the ratio of the area of a circle to that of a square on its diameter is $\pi : 4$. No new constant of proportionality is needed. Reversing the logic, the π in (2) is the same as that in (1). All is well. But not all is over, as we shall see next.

1.3 A new problem fuels many discoveries

The reappearance of π to also denote the constant of proportionality between the areas of a circle and the square on its radius generated a new problem known as the quadrature of the circle (or squaring the circle) problem:

> "There surely exists a square whose area equals πr^2, which is the area of a circle of radius r. Construct that square with a straightedge and compass."

The problem is equivalent to drawing a line segment of length $\sqrt{\pi}$, given a unit length. As innocuous as it looks, a quest to solve this problem has resulted in much advancement in mathematics. More than two millennia after Archimedes, in 1882, Ferdinand von Lindemann solved the problem *in the negative* by showing that π is a transcendental number (not algebraic; that is, not the root of a finite-degree polynomial with integer coefficients); as such, it cannot be constructed with a straightedge and compass. About the transcendence of π, more in Subsection 2.2. Stay tuned.

Lambert (1761) proved that π is an irrational number (not a ratio of two integers) by using continued fraction (CF) representation of $\tan x$. For a modern treatment of Lambert's CF for $\tan x$, see the free online book [10], Sections 20 and 21 of Chapter 34. For a simple proof of the irrationality of π, see [22]. However, to this day, it is not known whether or not π is a normal number (in which all possible n-tuples of digits occur equally often for all finite n). See [32] for some evidence and [2] for a more recent update.

1.4 Constant ratios in higher dimensions

Having studied various measurements in a circle, both perimeter and area, naturally, we wish to extend the measurements to the three-dimensional (3-D) sphere. How did π show up in the surface area of a sphere of radius R given below?

$$S = 4\pi R^2. \tag{3}$$

You might say, "Why not? After all, a sphere is obtained by revolving a (semi-)circle about the diameter." I am delighted you were anticipating π in formula (3). And congratulations! Your hunch is correct. But, momentarily playing the Devil's advocate (Just for the record, I am acting in this capacity *pro bono.*), let me counter, "If so, why don't we see a π^2, or a $\pi^{3/2}$, or some other power of π? Why is the power exactly one?"

The appearance of π in (3) can be explained using Euclidean plane geometry! Long before calculus was invented, Archimedes (287–212/211 BCE) used similar triangles and a limiting argument (by slicing a sphere thinly with planes parallel to the equator) to show that the portion of the surface area of a sphere of radius R sandwiched between two successive parallel planes *equals* the corresponding portion of the lateral surface area of a circumscribing cylinder (of radius R and height $2R$) sandwiched between those two parallel planes. See Figure 2, where $x : R = \Delta : l$; or equivalently, $x\,l = R\Delta$. Thereafter, by aggregation, the surface area of the sphere equals the lateral surface area of the circumscribing cylinder. While the former cannot be unfolded into a plane (such effort causes the Cartographer's dilemma, see [23]), the latter surface can be unfolded without distortion into a rectangle with side lengths $2\pi R$ and $2R$; hence, the area is $4\pi R^2$. See [29] for more details.

The Greek historian Plutarch (early 2nd century CE) relates that Archimedes requested the figure of a sphere with circumscribing cylinder be engraved on his tombstone, which is confirmed

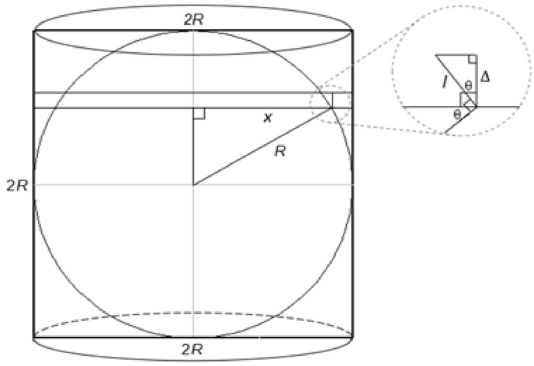

Figure 2: Archimedes explains the surface area of a sphere equals the curved surface area of the circumscribing cylinder using thin slices parallel to the equator.

by the Roman writer Cicero, who located the tomb in 75 BCE when he was quaestor of Sicily. I have begun to think ... "What do I want on my tombstone?" By the time you finish reading this paper, you may learn my wish.

Next, let us find a formula for the volume of a sphere of radius R. Take any *small* region of area A on the sphere and join all points on (the boundary of) the region to the center. The cone so formed is approximately a pyramid on a plane base of area A and height R; hence, its volume is one-third base times height or $(1/3)AR$. By aggregating such volumes as the base slides to span the entire surface of the sphere, one can obtain the volume of a sphere of radius R to be

$$V = \frac{1}{3} S\, R = \frac{1}{3} 4\pi R^2\, R = \frac{4}{3}\pi R^3. \tag{4}$$

Again, there is no surprise in the presence of π (and not a different power of π) in (4).

However, why stop at dimension 3? We, who live in the post-calculus era, can find the surface area of an n-dimensional (n-D) unit hyper-sphere (see details in [34]) as

$$S_n = \frac{\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^n}{\int_0^{\infty} e^{-r^2} r^{n-1} dr} = \frac{\sqrt{\pi}^n}{\frac{1}{2}\Gamma(n/2)} = \frac{2\pi^{n/2}}{\Gamma(n/2)} = \begin{cases} \frac{2^n \pi^{(n-1)/2}((n-1)/2)!}{(n-1)!} & \text{for odd } n, \\ \frac{2\pi^{n/2}}{(n/2-1)!} & \text{for even } n \end{cases} \tag{5}$$

and the volume of an n-D unit hyper-sphere as $V_n = \int_0^1 S_n r^{n-1} dr = S_n/n$. The surface area and the volume can be obtained recursively as

$$S_{n+2} = \frac{2\pi}{n} S_n \quad \text{and} \quad V_{n+2} = \frac{2\pi}{n+2} V_n.$$

In particular, the surface and the volume of a 4-D and a 5-D hyper-spheres involve π^2, those of 6-D and 7-D hyper-spheres involve π^3, etc. So, it was not for nothing that I played the Devil's Advocate a while ago. I rest my case.

Interested readers can check from (5) that the surface area of a unit n-sphere increases as n ranges from 1 to 7, reaching a maximum for $n = 7$, and thereafter the surface area decreases to 0 as $n \to \infty$. Likewise, the volume of a unit n-sphere, $V_n = S_n/n$, increases as n ranges from 1 to 5, reaching a maximum for $n = 5$, and thereafter, the volume decreases to zero as $n \to \infty$. See [34]. We will revisit n-spheres towards the end of this paper.

2 Anticipated Presence of π

Several alternative expressions of π were proposed to compute π efficiently. As mathematics advanced, new sub-disciplines arose, and new techniques emerged during the last half millennium. Since the target is π or π^{-1}, its presence is well anticipated.

2.1 Computing π via sequences, series, and infinite products

Ready for a quiz? Guess the limit of the following sequence involving n nested radicals

$$h_n = 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}. \tag{6}$$

as $n \to \infty$. Please note the single minus sign in (6). The limit of (6) will become obvious once we disclose how Courant (1888-1972) and Robbins (1915-2001) constructed it. See [11], p. 124. Inscribe a regular 2-gon (just a diameter); then bisect the angles between successive vertices of a regular 2-gon to form a regular $2^2 = 4$-gon (an inscribed square). Then bisect the angles between successive vertices of the regular 2^2-gon to form a regular 2^3-gon (an inscribed octagon). Then bisect again to form a regular 2^4-gon (an inscribed hexadecagon). Etc.

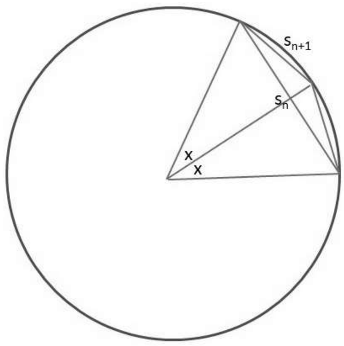

Figure 3: Length of a side of a 2^{n+1}-gon as a function of that of a 2^n-gon

Suppose that s_n denotes the length of a side of a regular 2^n-gon. Then $s_1 = 2, s_2 = \sqrt{2}$, and thereafter, a recursive relation (see Figure 3) holds:

$$s_{n+1} = \sqrt{(s_n/2)^2 + [1 - \sqrt{1 - (s_n/2)^2}]^2} = \sqrt{2 - 2\sqrt{1 - (s_n/2)^2}} = \sqrt{2 - \sqrt{4 - s_n^2}}.$$

Solving the recursive relation, we get

$$s_3 = \sqrt{2 - \sqrt{2}}, \quad s_4 = \sqrt{2 - \sqrt{2 + \sqrt{2}}};$$

and, in general, by mathematical induction on $n \geq 1$, we have

$$s_{n+1} = \underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}}_{\text{involves } n \text{ radicals}}.$$

Note that the perimeter of a regular 2^{n+1}-gon inscribed in a unit circle is $2^{n+1}s_{n+1}$, which monotonically increases to 2π as $n \to \infty$, implying that $\lim_{n\to\infty} 2^n s_{n+1} = \lim_{n\to\infty} h_n = \pi$. Had you read the title of this subsection with care, you would have known the answer.

Let us look at some series that converge to π or its transformation. By evaluating the Gregory–Leibniz series (1674)

$$\tan^{-1} x = \int_0^x \frac{1}{(1+u^2)} du = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \tag{7}$$

at $x = 1$, one gets

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \cdots$$

See [11], p. 441–442. As elegant as this formula is, the convergence is extremely slow: Five billion terms are required for the Gregory-Leibniz series to compute π correct to 10 decimal places. A faster convergence was already obtained by Keḻallur Nilakantha Somayaji (1444–1544)

$$\pi = 3 + \frac{4}{2\times3\times4} - \frac{4}{4\times5\times6} + \frac{4}{6\times7\times8} - \frac{4}{8\times9\times10} + \cdots \tag{8}$$

which holds because

$$\begin{aligned}
&\frac{1}{2\times3\times4} - \frac{1}{4\times5\times6} + \frac{1}{6\times7\times8} - \frac{1}{8\times9\times10} + \cdots \\
&= \left(\frac{1}{2\times2} - \frac{1}{3} + \frac{1}{2\times4}\right) - \left(\frac{1}{2\times4} - \frac{1}{5} + \frac{1}{2\times6}\right) + \left(\frac{1}{2\times6} - \frac{1}{7} + \frac{1}{2\times8}\right) - \cdots \\
&= \frac{1}{4} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi-3}{4}.
\end{aligned}$$

Nonetheless, in 1705, Abraham Sharp (1651-1742) substituted $x = \sqrt{1/3}$ in (7) to obtain

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3\cdot3} + \frac{1}{3^2\cdot5} - \frac{1}{3^3\cdot7} + \frac{1}{3^4\cdot9} - \dots\right).$$

In 1706, starting from $\beta = \tan^{-1}(1/5)$ and using the tangent of double angle formula twice, John Machin (1680-1752) obtained $\tan(4\beta) = 120/119 = (1+1/239)/(1-1/239) = \tan(A+B)$; or equivalently, $4\tan^{-1}(1/5) = 4\beta = A + B = \pi/4 + \tan^{-1}(1/239)$, or

$$\frac{\pi}{4} = 4\left(\frac{1}{5} - \frac{1}{3\cdot5^3} + \frac{1}{5\cdot5^5} - \dots\right) - \left(\frac{1}{239} - \frac{1}{3\cdot239^3} + \frac{1}{5\cdot239^5} - \dots\right).$$

In 1844, Johann Martin Zacharias Dase (1824–1861) calculated π to 200 decimal places in two months, a record for the time, from the Machin-like formula

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}.$$

A faster formula $\pi/4 = 6\tan^{-1}(1/8) + 2\tan^{-1}(1/57) + \tan^{-1}(1/239)$ was found by Stormer in 1896, and was used in 1961 by Shanks and Wrench (at the IBM Data Processing Center, New York) to compute π to a little over 100,000 decimal places.

In 1593, the French mathematician François Viète (1540–1603) began with the area $A_2 = 2$ of a square (a regular 2^2-gon) inscribed in a unit circle, with each side subtending at the center an angle $\beta_2 = 2\pi/2^2$; then he successively doubled the number of sides of the inscribed polygon (hence halved the central angle to become $\beta_i = 2\pi/2^i$); he noted that $A_3 = A_2/\cos\beta_3$, $A_4 = A_3/\cos\beta_4 = A_2/[\cos\beta_3 \cdot \cos\beta_4], \ldots$, $A_k = A_2/[\cos\beta_3 \cdot \cos(\beta_4) \cdot \ldots \cos(\beta_k)]$. Thereafter, using $\cos(\theta/2) = \sqrt{1/2 + (1/2)\cos\theta}$, and taking limit as $k \to \infty$, he obtained

$$\begin{aligned} \frac{2}{\pi} &= \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdots ; \\ \pi &= \frac{2}{1} \cdot \frac{2}{\sqrt{2}} \cdot \frac{2}{\sqrt{2+\sqrt{2}}} \cdot \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \cdots . \end{aligned} \tag{9}$$

Both (9) and (6) use the same construction; but (9) computes the area and (6) computes the perimeter of the 2^n-gon. With some algebraic manipulations, interested readers can verify that these two expressions are identical! That is, for all $n \geq 1$, the n-th approximants of π obtained from these two expressions are the same.

The English clergyman and mathematician John Wallis (1616–1703) in 1655 found

$$\frac{\pi}{2} = \left(\frac{2}{1} \cdot \frac{2}{3}\right) \cdot \left(\frac{4}{3} \cdot \frac{4}{5}\right) \cdot \left(\frac{6}{5} \cdot \frac{6}{7}\right) \cdot \left(\frac{8}{7} \cdot \frac{8}{9}\right) \cdots .$$

We cannot report here how Wallis derived it, before there was integral calculus. See [33]. But we should point out that his formula requires only rational operations (no need to extract roots) to compute π. Today one can use integration by parts to show that

$$\int_0^{\pi/2} \sin^{2m} x \, dx = \frac{1 \cdot 3 \cdot 5 \ldots (2m-1)}{2 \cdot 4 \cdot 6 \ldots (2m)} \frac{\pi}{2} ; \quad \int_0^{\pi/2} \sin^{2m+1} x \, dx = \frac{2 \cdot 4 \cdot 6 \ldots (2m)}{1 \cdot 3 \cdot 5 \ldots (2m+1)} .$$

Since the ratio of the two integrals tends to one as $m \to \infty$, we get Wallis's formula. Let us count our blessing for living in the post-calculus era.

Sir Isaac Newton (1642–1727), who shares credit with Gottfried Wilhelm Leibniz (1646–1716) for developing infinitesimal calculus, used the binomial theorem to obtain

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \ldots,$$

which upon integration becomes

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \ldots;$$

and on evaluation at $x = 1/2$, yields

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot \cdot 6 \cdot 7 \cdot 2^7} + \ldots .$$

During the Plague years 1665-6, "$\cdots$ having no other business at the time", Newton considered a semi-circle with diameter (0,1) and radius 1/2. See Figure 4. The area a of the half-circular segment on $(0, 1/4)$ equals one-sixth of the circle minus one-half of an equilateral triangle of side length 1/2, or $\pi/24 - \sqrt{3}/32$. But it also equals the integral $\int_0^{1/4} \sqrt{x - x^2}\, dx =$

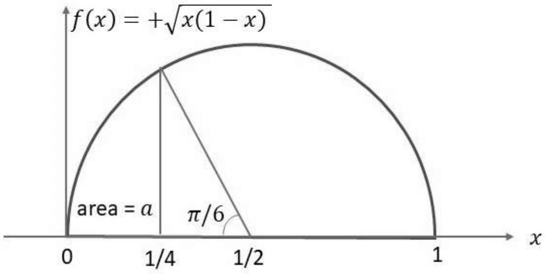

Figure 4: Newton integrates a half-segment of a circle.

$\int_0^{1/4}(\sqrt{x}\cdot\sqrt{1-x})\,dx$. The binomial theorem expansion of $\sqrt{1-x}$ into an infinite series, times $\sqrt{x}$, when integrated term by term yields

$$\pi = \frac{3\sqrt{3}}{4} + 2 - 24\left(\frac{1}{5\cdot 2^5} + \frac{1}{28\cdot 2^7} + \frac{1}{72\cdot 2^9} + \dots\right).$$

The story of π will remain incomplete if we do not mention "The Man Who Knew Infinity", Srinivasa Ramanujan (1887–1920), who published dozens of innovative, elegant, deep, and fast-converging formulas for π. Here are three such formulas (see [38]), of which the first one was already proved by Gustav Conrad Bauer in 1859; but who could have the faintest idea about the latter two? Please do not ask me how the magician did it.

$$\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1\times 3}{2\times 4}\right)^3 - 13\left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)^3 + \cdots.$$

$$\frac{2\sqrt{2}}{\sqrt{\pi}\,\Gamma^2\left(\frac{3}{4}\right)} = 1 + 9\left(\frac{1}{4}\right)^4 + 17\left(\frac{1\times 5}{4\times 8}\right)^4 + 25\left(\frac{1\times 5\times 9}{4\times 8\times 12}\right)^4 + \cdots.$$

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}.$$

In response to Hardy's curiosity "How does it come into your mind?" Ramanujan was lost for an answer: "I don't know; it just happens; it's like a revelation; it just comes floating in. It must be God; it has to be the mind of God engaging with me."

Inspired by the last formula, which yields eight digits per term, American brothers David Volfovich Chudnovsky (1947–) and Gregory Volfovich Chudnovsky (1952–) developed in 1987 the Chudnovsky algorithm (see [9]) used nowadays to calculate the digits of π with extreme precision. Modern calculations of π use the Chudnovsky algorithm. About 14 additional terms are found by evaluating each term in the expression below:

$$\frac{1}{\pi} = \frac{12}{640320^{3/2}}\sum_{k=0}^{\infty}\frac{(6k)!(13591409+545140134k)}{(3k)!\,k!^3(-640320)^{3k}}.$$

2.2 Alternative definitions of π

As a precursor to defining π using complex numbers, we must recall the base of natural logarithm $e = \lim_{n\to\infty}(1+1/n)^n \approx 2.7318281828459\cdots$. It was discovered in 1683 by Jacob Bernoulli (1655-1705) while he was studying compound interest. It is called Euler's number in honor of

Leonhard Euler (1707–1783) who placed his indelible signature on it by doing three things: (1) In 1731 he first introduced the symbol e and proposed its ideal fit as the base of Napier logarithm (hence, it is also called Napier's constant). (2) He proved that $e = \sum_{k=0}^{\infty}(1/k!)$ by evaluating at $x = 1$ Taylor's series expansion of the inverse-logarithm (or exponential) function e^x whose derivative is itself. (3) In 1737, he proved that e is irrational using its CF representation.

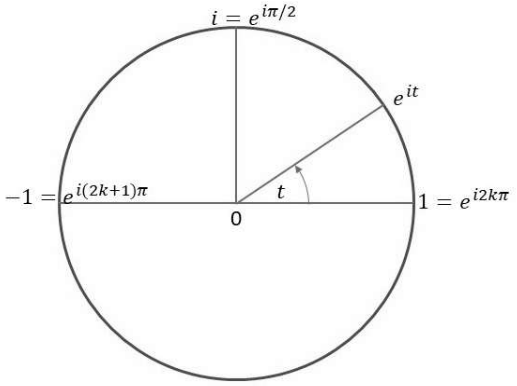

Figure 5: Complex numbers of the form e^{it}

Define π via complex roots of 1: All complex-number solutions to $e^z = 1$ are in (imaginary) arithmetic progression of the form $\{i\,2t\,k : k = 0, \pm1, \pm2, \ldots\}$. See Figure 5.

$$\text{The unique positive real number } t \text{ such that } e^{i2t} = 1 \text{ is defined to be } \pi. \tag{10}$$

Euler's formula, $e^{i\phi} = \cos\phi + i\sin\phi$, implies that $e^{i\,2\pi} = 1$ and $e^{i\pi} + 1 = 0$. The latter, known as Euler's identity, has been called "the most remarkable formula in mathematics" by Richard P. Feynman, for its single uses of the notions of addition, multiplication, exponentiation, and equality, and the single uses of the important constants $0, 1, e, i, \pi$.

In 1873, Charles Hermite (1821–1901) proved that e is a transcendental number using continued fraction (CF) representation. Lindemann (1882) proved that if a complex number z is algebraic then $e^z + 1$ cannot be 0. But Euler's identity says that $e^{i\pi} + 1 = 0$. Hence, $i\pi$ is transcendental, and so is π. Lindemann's proof was later simplified by other mathematicians. See [17] or [21].

In a 1988 (unscientific) survey, *Mathematical Intelligencer* asked its readers to rank 24 theorems on a scale of 0–10, for beauty. Euler's identity was ranked the first, the transcendence of π the eighth, and Nilakantha's series (8) for π the fourteenth. See [35].

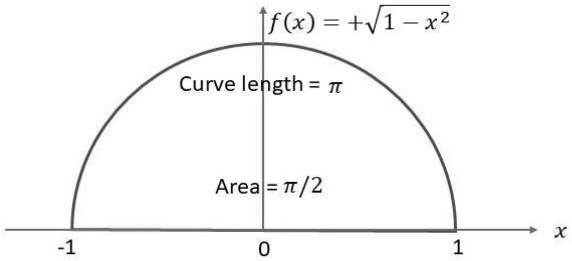

Figure 6: Arc length and area of a semi-circle

In the nineteenth century, attempts were made to define π free of the circle. See Figure 6. In 1841, starting from the functional form of the upper semi-circle, $f(x) = \sqrt{1-x^2}$, $x \in [-1, 1]$, Karl Weierstrass applied the arc-length formula for a curve to obtain π as

$$\int_{-1}^{1} \sqrt{1+(f'(x))^2}\,dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = 2\int_{0}^{1} \frac{1}{\sqrt{1-x^2}}\,dx. \tag{11}$$

Alternatively, using the area under the upper semi-circle, π could be defined as

$$\pi = 2\int_{-1}^{1} \sqrt{1-x^2}\,dx = 4\int_{0}^{1} \sqrt{1-x^2}\,dx. \tag{12}$$

However, since in university curriculum differential calculus precedes integral calculus, in 1870, Richard Baltzer defined π as **twice** the smallest positive root of

$$1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots = 0. \tag{13}$$

Attentive readers will note that the left-hand side of (13) is Taylor's series expansion of $\cos t$, which admits as the smallest positive root $\pi/2$.

Formulas (10)–(13) are post-calculus alternative definitions of π without direct reference to the circle or the n-spheres.

The expressions presented in this section were obtained keeping π or π^{-1} in focus. In the next section, we present expressions that arose in unrelated contexts. In each example, ask yourself: "Where did π come from?" Then try to answer the question if you can.

3 Unexpected Appearances of π

In the seventeenth century, with the discovery of many other curves, refinement of algebra and trigonometry, and discovery of calculus, π was freed from the circle (sphere and hyper-spheres). Today π appears in many branches of mathematics, such as number theory, probability, analysis, etc., and also in physics and engineering.

3.1 Archimedean Algorithm

The paper [6], commemorating the 2015 centennial of the Mathematical Association of America, mentions that Pfaff–Borchardt–Schwab discovered severally during the nineteenth century a mean iteration technique to compute π: Set $a_0 = 2\sqrt{3} = 3.464102\cdots$ and $b_0 = 3$. Then let $a_{n+1} = 2a_nb_n/(a_n+b_n)$ be the harmonic mean, and $b_{n+1} = \sqrt{a_{n+1}b_n}$ be the geometric mean. Below we show (to six decimal places) the first few iterations:

[a] 3.215390 3.159660 3.146086 3.142715 3.141873 3.141663 3.141610 3.141597 3.141594 3.141593

[b] 3.105829 3.132629 3.139350 3.141032 3.141452 3.141558 3.141584 3.141590 3.141592 3.141593

The inter-twined sequences a_n and b_n converge to π, with the error decreasing by a factor of four with each iteration. See [5]. How does π appear in the limit?

No trigonometric function has been used in this algorithm; hence it can stand alone without reference to the circle. However, the proof of the convergence to π is related to the approximation of the half-perimeter of a unit circle inspired by Archimedes's evaluation of the perimeters of regular n-gons for $n = 6, 12, 24, 48, 96, \ldots$. Note that a_0 (and b_0) are the half-perimeters of the circumscribed (and inscribed) regular hexagons about (and in) a unit circle. Now that the secret is revealed, it should not come as a big surprise that [24] derives Viète's formula starting from the Archimedean algorithm since both procedures involve successive doubling of the number of sides of an inscribed m-gon.

3.2 Basel Problem

In 1735 Euler solved the Basel problem (posed in 1650 by Pietro Mengoli) of finding the sum of the reciprocal squares (see [11], p. 509–510, 482):

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}. \tag{14}$$

The left-hand side of (14) is the sum of inverse squares of natural numbers, with no apparent connection to the circle, and often denoted by $\zeta(2)$ because it is the Riemann Zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ evaluated at 2. Yet on the right-hand side π^2 flashes in like a bolt from the blue. For a proof of (14), see [30], or [28] p. 284, or [13]. The result is both surprising and beautiful (it ranked fifth in [35]). Therefore, I must beg your permission to present a synopsis of my favorite proof found in [25]. But please do not insist on it right now. Instead, wait until Subsection 3.3.

Incidentally, this result proves that a randomly chosen natural number is square-free (not divisible by a perfect square other than 1) or two randomly chosen natural numbers are relatively prime (have no common factor other than 1) with probability $\xi = 6/\pi^2$. Here is the reason: For every prime p, the selected natural number must not be divisible by p^2, and at least one of the two selected natural numbers must not be divisible by p. Each event happens with probability $1 - p^{-2}$. Hence,

$$\xi = \prod_{p \text{ prime}} (1 - p^{-2}) = \left(\prod_{p \text{ prime}} \frac{1}{1 - p^{-2}} \right)^{-1} = \left(\prod_{p \text{ prime}} [1 + p^{-2} + p^{-4} + p^{-6} + \ldots] \right)^{-1}$$
$$= \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \right)^{-1} = \frac{6}{\pi^2} = .607927 \cdots .$$

where, in the last line, we have used the Fundamental Theorem of Arithmetic (also known as the Unique Prime Factorization Theorem) to factor every natural number.

3.3 Probability distributions

For what choice of the constant k is the function $g(x) = k\ e^{-x^2/2}$ defined on $(-\infty, \infty)$ a genuine probability density function (PDF); that is, the integral equals one? With the choice of $k = 1/\sqrt{2\pi}$, the test function $g(x)$ becomes the PDF of the standard normal distribution (see Figure 7), also known as the Gaussian distribution in honor of Johann Carl Friedrich Gauss (1777–1855). How did π find a place in this distribution?

To explain, we must prove that the integral $I = \int_{-\infty}^{\infty} e^{-x^2/2} dx$ evaluates to $\sqrt{2\pi}$. This integral already appeared (in slight disguise) in formula (5) for the surface area of a hyper-sphere. To

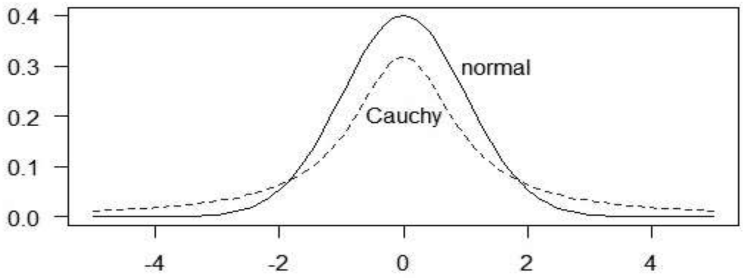

Figure 7: Standard normal and standard Cauchy distributions

prove it, we can compute its square using polar transformation: $x = r\cos\theta, y = r\sin\theta$, with Jacobian r. Then we obtain

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\cdot r\,dr\;d\theta = 2\pi\int_0^{\infty} e^{-u}du = 2\pi.$$

The calculation above also shows that if X, Y are independent standard normal variables, then $W = (X^2 + Y^2)/2 = r^2/2$ has the standard exponential distribution with PDF e^{-w} on $(0, \infty)$; and independently of W, the transformation $\Theta = \tan^{-1}(Y/X)$ is a random angle having a uniform $(0, 2\pi)$ distribution.

Do you now see the relation to the circle? When X and Y are independent, normally distributed variables, the joint PDF of (X, Y) is the product of the marginal PDF's. Hence, it has contour plots (set of points with the same PDF value) that are concentric circles. No wonder π claims its rightful place in the standard normal PDF.

Let us look at another example. For what choice of the constant c is the function $h(x) = c\,(1 + x^2)^{-1}$ defined on $(-\infty, \infty)$ a genuine PDF? This time, choose $c = 1/\pi$ to obtain the PDF of the standard Cauchy distribution (see Figure 7), named after Augustin-Louis Cauchy (1789–1857). To explain how π entered this scene, note that the derivative of $\tan^{-1} x$ is $(1 + x^2)^{-1}$. Hence,

$$\int_{-\infty}^{\infty}(1 + x^2)^{-1}dx = \tan^{-1} x|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

How is the standard Cauchy distribution connected to the circle?

Let me tell you a story: Imagine two parallel, infinitely long walls two units apart. A rotating jugs machine (or a pitching/bowling machine) placed midway between the walls throws (a large number of) balls in random directions distributed uniformly over $(0, 2\pi)$. See Figure 8. Assuming that the balls travel straight after ejection, the histograms of points where the balls hit either wall for the first time resemble the standard Cauchy distribution, exhibiting the connection to the circle. Stripping the story, we may say: If Θ has a uniform $(0, 2\pi)$ distribution, then $\tan\Theta$ has a standard Cauchy distribution.

A direct (and hopefully no longer intriguing) connection between the above two examples is that the standard Cauchy variable X is the ratio of two independent standard normal variables. Consequently, the PDF's of X and X^{-1} are the same, implying that $\int_0^1(1 + x^2)^{-1}dx = \pi/4$.

In that same spirit, let us ask: What is the PDF of the ratio Y of two independent, folded standard Cauchy variables, or of $Y = |X_1|/|X_2|$, where the X_i's are independent standard Cauchy variables?

Using routine change of variables $y = x_1/x_2$ and $w = x_2$, with Jacobian w, and integrating

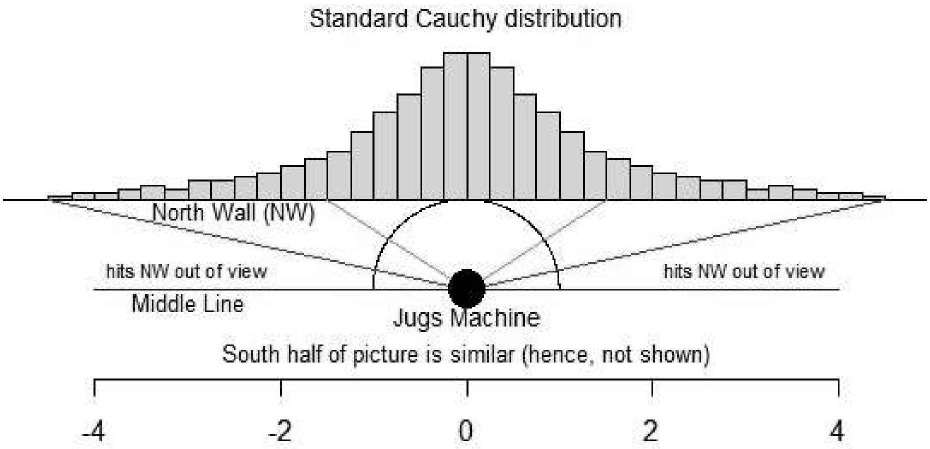

Figure 8: A jugs machine throws balls in a uniformly distributed angular direction. Balls travel straight and hit one of the walls. The points of impact (on each wall) follow a standard Cauchy distribution. About 12.6% of balls that hit the north wall, hit it outside $[-5, 5]$.

out w, we obtain the PDF of Y as $\frac{-4\ln y}{\pi^2(1-y^2)}$ on $(0,\infty)$. By definition, the PDF's of Y and Y^{-1} are the same, implying that

$$\int_0^1 \frac{-\ln y}{(1-y^2)}\,dy = \frac{\pi^2}{8}.$$

Thereafter, expanding $(1-y^2)^{-1} = 1+y^2+y^4+\ldots$, changing variable $u = -\ln y$, and integrating term by term, the last integral equals $\int_0^\infty u\,(e^{-u}+e^{-3u}+e^{-5u}+\ldots)\,du$, or the sum of inverse squares of odd numbers. Adding to it the sum of inverse squares of even numbers, we obtain

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \sum_{j=1}^{\infty}\frac{1}{(2j-1)^2} + \sum_{j=1}^{\infty}\frac{1}{(2j)^2} = \frac{\pi^2}{8} + \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}, \text{ which implies } \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

This is the synopsis of the proof in [25] of Euler's solution to the Basel problem.

3.4 Large factorials

Large factorials must be evaluated when counting arrangements of gas particles, quantum particles in different macro states, or files in data compression. Quite unexpectedly, π appears in Stirling's (1730) approximation $n! \sim \sqrt{2\pi n}(n/e)^n$, where '$\sim$' means the ratio of the two sides converges to 1, as $n \to \infty$. The convergence is very fast: While $10! = 3\,628\,800$, the approximate value is $3\,598\,696$, which is only a 0.83% underestimate. Similarly, 100! is underestimated by a mere 0.083%.

This asymptotic formula is named after James Stirling (1692–1770), though it was stated earlier by Abraham de Moivre (1667–1754), except for the constant $\sqrt{2\pi}$. Here is a heuristic idea of a proof that uses Laplace's method (1774) for approximation, see details in [37], discovered by Pierre-Simon, Marquis de Laplace (1749–1827):

$$\begin{aligned} n! = \Gamma(n+1) &= \int_0^\infty x^n e^{-x}\,dx = \int_0^\infty e^{n\ln x - x}\,dx \\ &\sim e^{n\ln n - n}\int_{-\infty}^{\infty} e^{-\frac{(y-n)^2}{2n}}\,dy = \left(\frac{n}{e}\right)^n\sqrt{2\pi n}. \end{aligned} \tag{15}$$

We may rewrite (15) to obtain π as the limit of a sequence involving factorials and exponentials

$$\lim_{n\to\infty} \frac{e^{2n} n!^2}{2n^{2n+1}} = \pi.$$

3.5 Asymptotic formula for the number of partitions

The partition function $p(n)$ is the number of ways a positive integer can be written as a sum of strictly positive integers. For example, $p(4) = 5$ because 4 can be written in five ways: $1+1+1+1; 1+1+2; 1+3; 2+2; 4$. Similarly, $p(9) = 30, p(14) = 135, p(19) = 490, p(204) = 5\,590\,088\,317\,495$. Interested readers may use the free WolframAlpha Calculator to find $p(n)$ by submitting the code: partition(204).

Littlewood [20], in his review of Ramanujan's collected works, calls the following a result of "supreme beauty":

$$\frac{5[(1-x^5)(1-x^{10})(1-x^{15})\cdots]^5}{[(1-x)(1-x^2)(1-x^3)\cdots]^6} = p(4) + p(9)x + p(14)x^2 + p(19)x^3 + \cdots \tag{16}$$

"But where is π in (16)?" you might ask. Thank you for staying alert. You will presently see π in the approximation of $p(n)$, for large n.

The partition function $p(n)$ is the coefficient of x^n in $\prod_{i=1}^{n}(1-x^i)^{-1}$. In 1918, Godfrey Harlod Hardy (1877–1947) and Srinivas Ramanujan (1887–1920) proved what we call today the Hardy-Ramanujan Asymptotic Partition Formula: As $n \to \infty$,

$$p(n) \sim \frac{1}{4n\sqrt{3}}\, e^{\pi\sqrt{2n/3}}. \tag{17}$$

The right hand side of (17) evaluated at $n = 204$ gives $5\,767\,332\,435\,055$, which is a 3% overestimate. But when evaluated at $n = 2004$, it is only a 1% overestimate. The percentage error drops as n increases. Even though $p(n)$ is a pure number theory concept with no connection to the circle, how does π appear in this asymptotic formula? I must step aside and let the originators explain. See [16]. Also, see [12] (and some references therein) for applications of this most beautiful formula in number theory. Rewriting (17), we obtain π as the limit of a sequence involving the partition function and logarithm

$$\lim_{n\to\infty} \frac{\ln[4n\sqrt{3}\, p(n)]}{\sqrt{2n/3}} = \pi.$$

3.6 Stochastic processes

In this subsection, we look at three stochastic processes. In the first one, π appears naturally. There is no surprise here; it serves as a warm-up exercise. In the second stochastic process, the appearance of π, though not immediately anticipated, can be explained with hindsight. The third stochastic process is too deep to anticipate the arrival of π in its answer or to explain its presence even when we know it has arrived.

Choose a pair of points (X, Y) at random in the square $[-1, 1] \times [-1, 1]$; call it a success if $X^2 + Y^2 \le 1$, and a failure otherwise. Then four times the proportion of success converges to π, as the number of chosen points increases to infinity. This exercise can teach students how many iterations are needed to reduce the error to a desired tolerance. Indeed, the convergence is extremely slow.

The second stochastic process is known as Buffon's Needle Problem, solved by Georges-Louis Leclerc, Comte de Buffon (1707–1788). On a hardwood floor, where the floor lines are parallel at gaps of d, drop a needle (or a toothpick to protect the floor) of length d after giving it a good spin. See Figure 9. Then the probability that the needle cuts a line is $2/\pi$. Where from comes π in the answer?

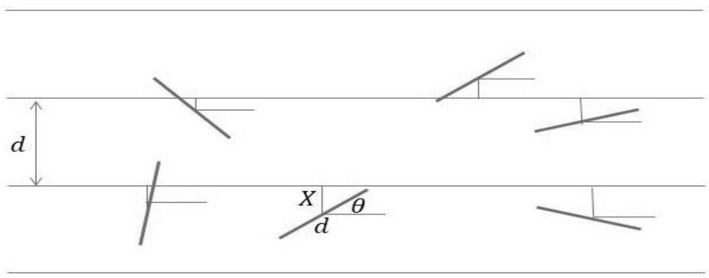

Figure 9: A randomly dropped spinning needle cuts a line with probability $2/\pi$.

A needle cuts a line if and only if X, the distance of the midpoint of the needle from the nearest line is at most $(d/2)\sin\Theta$, where Θ is the positive angle the needle makes with the positive direction of the floor lines. Since X is uniformly distributed over $(0, d/2)$, and independently of X, Θ is uniformly distributed over $(0, \pi)$, we have $(\Theta, Y = 2X/d)$ is uniformly distributed on the rectangle $(0, \pi) \times (0, 1)$. Therefore,

$$P\{X \leq (d/2)\sin\Theta\} = P\{Y \leq \sin\Theta\} = \int_0^\pi \frac{1}{\pi} P\{Y \leq \sin\theta\}\, d\theta = \frac{1}{\pi}\int_0^\pi \sin\theta\, d\theta = \frac{2}{\pi}.$$

Consequently, 2 over the proportion of times the needle cuts a line converges to π, as the number of drops increases to infinity. The connection to the circle is established by noting that the needle spins about its midpoint and stops in a random orientation that determines whether or not it intersects a floor line.

In the third stochastic process, π appears rather mysteriously: Flip n (a large number) fair coins; compute the absolute difference d between the numbers of heads and tails; repeat the process many times (using the same n coins) and compute the average $\overline{d}$ of the absolute differences. Then the ratio $2n/\overline{d}^2$ approaches π, as n increases. Where did this limiting value π come from? I am flabbergasted.

The mystery is solved using the central limit theorem (CLT), first published in 1733 by Abraham de Moivre and popularized by Pierre-Simon Laplace in 1812. It is (arguably) the Fundamental Theorem of Statistics because of three strong reasons: (1) It relates several concepts such as the sample size, the sample mean, the population mean, and the population standard deviation. (2) It holds for any population distribution with a finite variance. (3) It yields many statistical tests. See [31] for a modern-day proof that uses a moment-generating function approach (although one can adapt it to the characteristic function approach).

Since d is the absolute value of the sum of n independent random variables taking values in $\{-1, 1\}$ with probability $1/2$ each, by the CLT, $d/\sqrt{n}$ is distributed as an absolute standard normal variable $|Z|$, with expectation $E[|Z|] = \sqrt{2/\pi}$. Therefore, by the law of large numbers, $\overline{d}/\sqrt{n}$ converges in probability to $\sqrt{2/\pi}$. Equivalently, by Slutsky's theorem (1925), named after

Eugen Slutsky (1880–1948), $\overline{d}^2/(2n)$ (being a continuous function of $\overline{d}/\sqrt{n}$) converges to π^{-1}, and $2n/\overline{d}^2$ (being another continuous function) converges to π.

This third stochastic process is my advertisement for a graduate-level course in probability theory for mathematical statistics. There, many more wonders await you.

3.7 Physics

Letting g denote the earth's gravitational acceleration, the approximate period T of a simple pendulum of length L swinging with a *small* amplitude is given by

$$T \approx 2\pi\sqrt{\frac{L}{g}}. \tag{18}$$

Where from comes π in (18)? Is it because the pendulum traces the arc of a circle (of length L) as it swings back and forth? No. In fact, for a small amplitude, that arc is approximated by a line. The source of π is different.

To derive (18), proceed from Newton's Second Law of Motion: $F = ma$. For a pendulum with a small amplitude, the (linear) acceleration of the point mass is $-g\sin\theta \approx -g\theta$, and its angular acceleration is $\alpha \approx -g\theta/L$. Treating θ, the angular displacement of the pendulum from its central location as a function of time t, the angular acceleration is the second derivative of the angular displacement, or $\theta''(t)$. Thus, the oscillatory motion of the pendulum is a linear simple harmonic motion that satisfies the second-order linear differential equation

$$\theta''(t) + \frac{g}{L}\,\theta(t) = 0, \tag{19}$$

together with initial conditions: $\theta(0) = \theta_{\max}$ and $\theta'(0) = 0$. Notice that the differential equation cannot contain the mass of the pendulum bob (or it cancels out as a common factor). The solution to (19) is $\theta(t) = \theta_{\max}\cos\left(\sqrt{g/L}\;t\right)$, which is a periodic function with period given by (18) and is free of $\theta_{\max}$.

Since the period of the cosine function is 2π, its appearance in (18) is no longer mysterious. Even without knowing how to solve (19), the presence of 2π in (18) can be anticipated by interpreting the bob's actual location: It is the projection along a diameter of a base circle on which an imaginary bob moves at a uniform speed, while the string attached to the imaginary bob traces the slant face of a slender, right-circular cone. However, this explanation works for small amplitude only. By the way, if the string's length is chosen carefully so that $T = 2$ seconds, one can experimentally determine the value of g.

3.8 Geometry, but not of a circle

We conclude this section with the most stunning (in my opinion) appearance of π. In 1858 Rev. Hamnet Holditch (1800–1867), President of Gonville and Caius College in Cambridge, published the following geometrical theorem, which I first saw in [27].

Holditch's Theorem. Assume a chord can slide around a smooth, closed, convex curve C_1 with its two ends always on it. A fixed point, which splits the chord into two parts of lengths p, q, traces another closed curve C_2 within the first curve. Then the area between the two curves C_1 and C_2 is $\pi\,pq$.

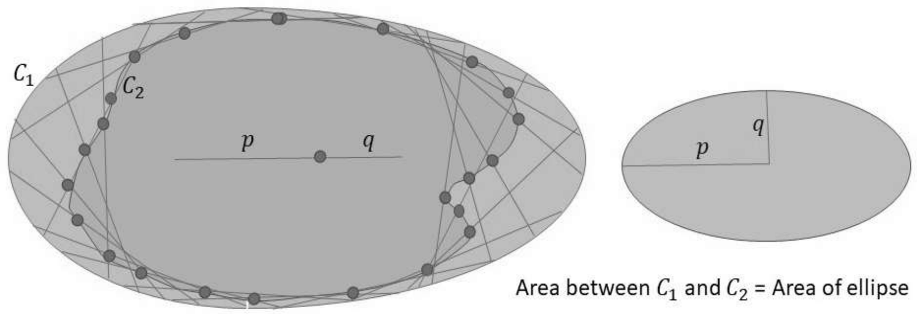

Figure 10: While a chord slides inside a smooth, closed, convex curve, a fixed point on the chord traces another curve. The area between the two curves is π times the product of the two segments into which the point splits the chord.

Holditch did not explicitly mention it, but one must also assume that the traced curve C_2 is simple (it does not intersect itself). Such a curve is also called a Jordan curve, named after Camille Jordan (1838–1922), who proved that every Jordan curve divides the plane into an "interior" region bounded by the curve and an "exterior" region containing all other points, so that every continuous path connecting a point in one region to a point in the other region intersects with the curve somewhere.

In the absence of modern day computer software such as *The Geometers' Sketchpad*, or even simple *PowerPoint*, which I have used to draw Figure 10, how did the reverend verify his discovery? He did not have to. He had a mathematical proof!

The fascinating aspect of Holditch's theorem is that the area between the given curve and the traced curve is independent of the shape and size of the given curve! Moreover, the area equals that of an ellipse with half-axes p and q. But there was no ellipse to begin with! The reverend's proof in [18], being short, understandable and freely accessible, is left to the reader.

As a special case, if the given curve C_1 is a circle of radius R, where $R > (p+q)/2$, then the traced curve C_2 is another concentric circle of radius $\sqrt{R^2 - pq}$, and the annular region has area $\pi\, pq$.

4 Unexpected Disappearances of π

We give several examples of the absence of π when we "naturally" anticipate its presence. Admittedly, "naturally" is a subjective word. Perhaps I should ask: What does your intuition tell you *a priori* about the presence or absence of π in a situation? How quickly will you revise your intuition after seeing a mathematical justification to the contrary?

4.1 Extract from a circle

Consider a sector of a unit circle casting a central angle $\pi/3$. (In fact, six such non-overlapping sectors cover the entire circle.) The sector consists of a unit equilateral triangle (ET) and a segment (seg) of the circle attached to one side of the ET. Attach such a segment to the other two sides so the sector becomes a unit Reuleaux triangle (RT). See Figure 11. Its perimeter is π; the area is $(\pi - \sqrt{3})/2$; and the width (or diameter) in every direction is 1. Thus, although not a circle, it is a 2-D object of constant width; its perimeter is the same as that of a circle of diameter 1, and its area is slightly smaller than that of the same circle.

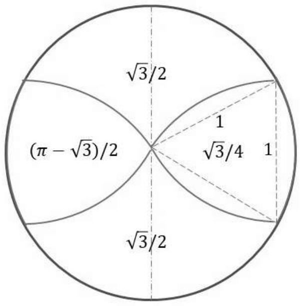

Figure 11: When two unit Reuleaux triangles are removed from a unit circle, the remaining area is free of π.

The unit RT can slide within the circle so that one of its "sides" always touches the circumference and the opposite "vertex" reaches the center. The shape of the remaining part of the circle does not change. Not very interesting. Also, you can fit three (but no more) such unit RTs inside the unit circle. They can slide within the circle; but their relative positions remain the same; and so do the uncovered three parts of the circle; each with three "sides" of length $\pi/3$ (one side is convex and two sides are concave) and area $(3\sqrt{3}-\pi)/6$. There is nothing much exciting here either.

But when you put two unit RTs inside the circle with one of their sides touching the circumference, their relative positions can change: The central angle θ between (the tangents to) the sides of the RTs varies between 0 and $2\pi/3$. However, when these two RTs are removed, the total area of the remaining two parts of the circle, connected only at the center, is a constant $\sqrt{3}$, free of π, for all θ!

"Who ate my pi?" would cry Baby Bear of *Goldilocks and the Three Bears*. We comfort Baby Bear by showing the following equalities in the area:

$$\text{circle} = 6 \text{ sectors} = 6 \text{ seg} + 6 \text{ ET} = (6 \text{ seg} + 2 \text{ ET}) + 4 \text{ ET} = 2 \text{ RT} + 4 \text{ ET}.$$

Moreover, π has not totally vanished: The total circumference of the remaining two parts is $8\pi/3$ for all θ. Also, each part has an area free of π if and only if $\theta = \pi/3$.

4.2 Extract from a cylinder

Next, consider the right circular cylinder $\{(x, y, z) : 0 \le x^2 + y^2 \le 1, 0 \le z \le H\}$ shown in Figure 12. It has radius 1, height H, two plane circular faces of perimeter 2π and area π, a curved surface of area $2\pi\, H$ and volume $\pi\, H$. So far, it is so good: Superstar π is in its rightful place. Merrily the world goes round and round.

Play a little mischief: Dissect this cylinder with a plane passing through points $(0, -1, 0)$, $(0, 1, 0), (1, 0, H)$. Then the smaller solid (which I wish will be inscribed on my tombstone) has a curved surface area $2H$ and volume $2H/3$, both are free of π! "Who Ate My Pi?" would complain Mama Bear.

To console Mama Bear, we explain: On the bottom semi-circular face of the smaller solid, draw several radii at small angular gaps. Slice the solid with vertical cuts along these radii.

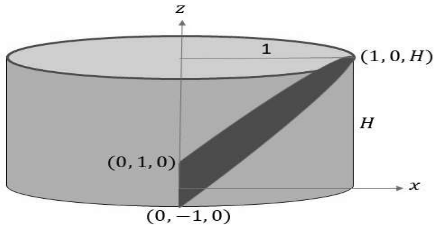

Figure 12: Slice a cylinder into two pieces using a slant plane passing through a diameter on one circular face and only one peripheral point on the opposite face.

As the slices fan out, remaining attached along the curved surface, the curved surface unfolds without distortion into a plane showing the graph of $H\,\sin\theta$ on $\theta \in (0, \pi)$, under which the area is $2H$; the center of the bottom face splits up to become the apex of every slice and remains at a distance one from the base on the new planar face. Aggregating the volumes of these thin pyramids, we find the volume of the smaller solid as $(1/3) \times 2H \times 1$. Nonetheless, π is still present in the areas of the semi-circular face $(\pi/2)$ and the semi-elliptical face $(\pi\sqrt{1+H^2}/2)$. Also, the bigger solid has three plane surfaces with areas $\pi/2, \pi\sqrt{1+H^2}/2, \pi$, one curved surface with area $2H(\pi - 1)$, and the volume is $H(\pi - 2/3)$.

4.3 Intersect identical cylinders

Suppose we have two identical cylinders with radius r and height $h = 2r$. Then each has volume $2\pi\, r^3$ and total surface area $6\pi\, r^2$. Suppose the cylinders intersect orthogonally (their axes intersect perpendicularly) at their centers. See Figure 13 (left panel). Then the 3-D region of intersection (region common to both cylinders) has volume $(16/3)r^3$ and surface area $16\, r^2$. "WHO ATE MY PI?" would growl Papa Bear.

Figure 13: Intersections, when two (left) or three (right) identical cylinders intersect (mutually) orthogonally, have volumes and surface areas free of π.

To calm Papa Bear down, use the Cavalieri Principle, named after Bonaventura Francesco Cavalieri (1598-1647). Imagine a sphere of radius r inscribed inside the intersection. Then slice the intersection with planes parallel to the plane containing the two axes leaving an infinitesimally small distance between successive planes. Notice that the cross-sections are squares with inscribed circles with their areas in the ratio $4 : \pi$. Accumulating these cross-sections, the volume of the intersection is $4/\pi$ times the volume of the sphere; or, $(4/\pi)\,(4/3)\pi\, r^3 = (16/3)\, r^3$, free of π. For surface area, use the above Cavalieri Principle and Archimedes's explanation for the surface area of a sphere mentioned in Section 1.

The only presence of π left in the solid of intersection (without cutting it further) is in the curved edges that form boundaries of two ellipses with half-axes $\sqrt{2}\, r$ and r, respectively. The total curve length is given by Colin Maclaurin's (1698–1746) expansion (see [8]) of the integral formula for the perimeter of an ellipse as

$$4\sqrt{2}\,\pi\, r\,\left[1-\sum_{k=1}^{\infty}\left(\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot 2k}\right)^2 \frac{1}{2^k(2k-1)}\right].$$

For our purpose, it suffices to know that the total edge exhibits π.

What if three such identical cylinders intersect mutually orthogonally? See Figure 13 (right panel). To learn how this intersection is easily created using either light beams or a cylindrical knife, see [4]. Then the 3-D region of intersection has volume $8\,(2-\sqrt{2})\,r^3$ and surface area $24\,(2-\sqrt{2})\,r^2$, both free of π. See [4] for justification and examples of many other intersections of cylinders where π is absent from volume and surface area.

4.4 Center of gravity

The center of gravity (CG) of a semi-circular lamina of uniform thickness, unit radius and made of material of uniform density is a point on the mid-radius line at a distance $\overline{x}_2$ from the midpoint of the diameter, where

$$\overline{x}_2=\frac{\int_0^1 x\cdot 2\sqrt{1-x^2}\,dx}{\int_0^1 2\sqrt{1-x^2}\,dx}=\frac{4}{3\pi}$$

involves π, just as one would naturally anticipate. But did you expect a π^{-1}? Note that the integral in the denominator is the area $\pi/2$ of a semi-circle and that in the numerator is the volume $2/3$ of the smaller slice cut off from the cylinder with $H=1$, that we discussed in Subsection 4.2.

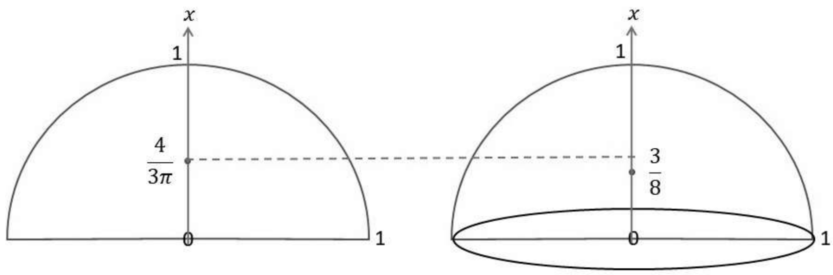

Figure 14: Center of gravity of a semi-circle and a hemi-sphere

Having shown the presence of π^{-1} in the CG of a semi-circle, we are puzzled when π vanishes from the CG of a hemisphere of unit radius and is made of material of uniform density. This CG is a point on the mid-radius line at a distance $\overline{x}_3=3/8$ from the center of the circular base, free of π! How come? It is because

$$\overline{x}_3=\frac{\int_0^1 x\,\pi\,(1-x^2)\,dx}{\int_0^1 \pi\,(1-x^2)\,dx}=\frac{\frac{1}{2}-\frac{1}{4}}{1-\frac{1}{3}}=\frac{3}{8}.$$

Note that π got canceled from the numerator and the denominator. But did you anticipate the CG would be free of π? I had to train my brain.

Are you sad to see π disappear from the CG of a hemisphere? Do not despair; carefully work out the CG of a 4-D hemi-hypersphere, which is $\overline{x}_4 = 16/(15\pi)$. See, π has returned in glory, albeit in its inverse incarnation. There is no stopping now: Check that $\overline{x}_5 = 5/16$, $\overline{x}_6 = 32/(35\pi)$, $\overline{x}_7 = 35/128$. Making a change of variable $x = \sin\theta$ and using integration by parts, you can show that for all $m \geq 2$,

$$\overline{x}_{2m+1} = \frac{2m+1}{2m+2}\,\overline{x}_{2m-1}; \quad \text{and} \quad \overline{x}_{2m+2} = \frac{2m+2}{2m+3}\,\overline{x}_{2m}.$$

Next, by mathematical induction, you can prove the alternate appearance and disappearance of π^{-1} in successive dimensions forever and ever! It reminds me of a magic show where a coin alternatively appears and disappears on the magician's palm.

4.5 Probability distributions

Suppose that Z has a standard normal distribution. Then the PDF of the folded standard normal variable $|Z|$ is $e^{-x^2/2}\sqrt{2/\pi}$ on $(0,\infty)$; and the PDF of the squared standard normal variable Z^2 (also known as a chi-square variable with one degree of freedom) is $e^{-x/2}/\sqrt{2\pi\, x}$ on $(0,\infty)$. Whereas $E[|Z|] = \sqrt{2/\pi}$ involves π as anticipated, why is $E[Z^2] = 1$, free of π? Caution: $E[Z^2]$ is neither $E^2[Z]$ nor $E^2[|Z|]$.

Next, suppose that Z_1, Z_2 are independent, standard normal variables. Then, as we saw in Subsection 3.3, Z_1/Z_2 has the standard Cauchy distribution with PDF $1/[\pi(1+x^2)]$ on $(-\infty,\infty)$, where π is present in the PDF. On the other hand, $W = Z_1^2 + Z_2^2$ (a chi-square variable with two degrees of freedom, or an exponential variable with mean 2) has PDF $(1/2)\,e^{-w/2}$ on $(0,\infty)$, and $R = \sqrt{Z_1^2 + Z_2^2}$ has PDF $r\,e^{-r^2/2}$ on $(0,\infty)$. Where did π go? On vacation to Madagascar? No, it has been absorbed into $\Theta = \tan^{-1}(Z_1/Z_2)$, which is stochastically independent of W (and hence of R) and has a uniform distribution on $(0, 2\pi)$.

The integral $I = \int_{-\infty}^{\infty} e^{-x^2/2}dx = \sqrt{2\pi}$ that we met in Subsection 3.3, helps us evaluate the gamma function $\Gamma(p) = \int_0^{\infty} x^{p-1}e^{-x}\,dx$ at positive half integers, because changing variable $x = u^2/2$, we evaluate

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} x^{-1/2}e^{-x}\,dx = \int_0^{\infty} \sqrt{2}e^{-u^2/2}\,du = \sqrt{2}\left(\frac{1}{2}\right)\sqrt{2\pi} = \sqrt{\pi}.$$

Next, using integration by parts, we see that

$$\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) \quad \text{for all } n \geq 1.$$

Thus, for any natural number n, we can see that $\Gamma(n+1/2)$ is a rational multiple of $\sqrt{\pi}$. However, again by integration by parts, we can see that $\Gamma(n) = (n-1)\Gamma(n-1)$, whence $\Gamma(n) = (n-1)!$ is an integer. Isn't it like a magic show where a bouquet alternately appears from and disappears into the magician's hat?

4.6 Acute triangles in high dimensions

We conclude this section with a mesmerizing truth, which I first came across in [27].

In 1982, Glen Richard Hall (1954–) posed and solved the acuteness of a random triangle: "Suppose that three points are selected independently and at random in an n-D ball (region enclosed by the n-D hyper-sphere). What is the probability P_n that the selected points form an acute triangle?"

The values of P_n illuminate the geometry of the n-D unit ball. Hall reported the exact formula for dimensions 2 and 3. Using "cumbersome, but essentially standard methods," Christian Buchta in 1986 gave the answers in closed form for all higher dimensions. Table 1 reports the formulas and values of P_n for $n \leq 9$.

Table 1: Probability P_n that three random points in an n-Ball form an acute triangle

P_n	exact formula	$\approx$ value
P_2	$4\pi^{-2} - 1/8$	0.280285
P_3	$33/70$	0.471429
P_4	$(256/45)\pi^{-2} + 1/32$	0.607655
P_5	$1415/2002$	0.706793
P_6	$(2048/315)\pi^{-2} + 31/256$	0.779842
P_7	$231161/277134$	0.834113
P_8	$(4194304/606375)\pi^{-2} + 89/512$	0.874668
P_9	$9615369/10623470$	0.905106

The probability of forming an acute triangle increases with dimension because three randomly chosen points in the n-D ball, when n is large, are highly likely to be close to mutually orthogonal unit vectors, implying that the triangle is almost an equilateral triangle of side length $\sqrt{2}$. But why does π^{-2} appear in the expressions for even dimensions, with a rational multiplier and a rational addendum. Why does it vanish for odd dimensions, leaving only a rational number? Moreover, why is the probability rational for odd dimensions, and why is there a rational additive constant on top of a rational multiple of π^{-2} for even dimensions?

To me, this result is like the grand finale in a magic show where a dove alternately appears and disappears according as the magician claps hands twice or thrice. How does the magician (and the dove) do it? I choose to remain silent for I am still awestruck. Inquisitive readers may read the papers [15] and [7].

Postlude

We have encountered some selected appearances of π in diverse areas of mathematics and physics, sometimes expected and other times unexpected, giving full or partial explanations and references to justify its presence. We have also wondered aloud why π turned out to be a no-show where we were anticipating its arrival — again with justifications and references.

Admittedly, the plot is too vast to be documented in just one mathematics paper. We sincerely hope our selections will whet the readers' appetite and inspire them to seek more appearances of π in both usual and strange places and wonder more about its unexpected absences elsewhere.

Acknowledgments. I wrote this paper in the Reader Expectation Approach of George Gopen. See [14]. I thank him heartily for teaching me this relatively new art. I especially thank my colleague Bogdan Nica for giving me feedback on an earlier draft leading to a better focus and some references where I found another unexpected appearance of π. Thanks to Debolina Chatterjee and Pratim Guha Niyogi, who proofread the paper and suggested Figure 8. This paper is an offering to Pastor Monica Miller, who, having provided spiritual nourishment, also live-streamed how to make and deck a home-baked pie (expected and appreciated) and then recited the first ten digits of π (not expected, but most appreciated). I hope to receive emails from my noble readers.

References

[1] Agarwal, Ravi P., Agarwal, Hans and Sen, Syamal K. (2013). Birth, growth and computation of pi to ten trillion digits. *Advances in Difference Equations* 2013:100. http://www.advancesindifferenceequations.com/content/2013/1/100

[2] Bailey, David H. and Borwein, Jonathan (2014). Pi Day is upon us again and we still do not know if Pi is normal. *Mathematical Monthly* 121, 191-206. The Mathematical Association of America.

[3] Beckmann, Petr (1971). *A History of π (PI).* Boulder, Colorado: The Golem Press, 1977; New York: Dorset Press, 1989.

[4] Bhattacharjee, Rishav and Sarkar, Jyotirmoy (2023). Strategic cuts with a cylindrical knife: Making solids bounded by cylindrical surfaces. *Resonance: The Indian Journal of Science Education*, **28(2)**, 293–312. https://www.ias.ac.in/article/fulltext/reso/028/02/0293-0312

[5] Borwein, Jonathan M. (2011). The life of Pi: From Archimedes to ENIAC and beyond. *Berggren Festschrift.*

[6] Borwein, Jonathan M. and Chapman, Scott T. (March 2015). I prefer Pi: A brief history and anthology of articles in the American Mathematical Monthly. *Amer. Math. Monthly* 122, 1–22. The Mathematical Association of America.

[7] Buchta, Christian, (1986). A note on the volume of a random polytope in a tetrahedron. *Illinois Journal of Mathematics*, 30(4), 653–659.

[8] MacLaurin, C. (1742) *A Treatise on Functions.* Edinburgh.

[9] Chudnovsky, DV, Chudnovsky, GV (1989). The computation of classical constants. *Proc. Natl. Acad. Sci.*, 86, 8178-8182.

[10] Chrystal, George. *Algebra: An Elementary Text-Book*, Adam & Charles Black, 1900. http://onlinebooks.library.upenn.edu/webbin/book/lookupid?key=olbp36404

[11] Courant, Richard and Robbins, Herbert (Revised by Stewart, Ian) (1996). *What is Mathematics?: An Elementary Approach to Ideas and Methods*, 2nd ed. Oxford University Press.

[12] Debnath, Lokenath (2015). A brief history of the most remarkable numbers π, g and δ in mathematical sciences with applications. *Int. J. Appl. Comput. Math* 1:607–638. DOI 10.1007/s40819-015-0038-6

[13] Eremenko, A. (2013). How Euler found the sum of reciprocal squares. https://www.math.purdue.edu/ eremenko/dvi/euler.pdf

[14] Gopen, George (2004). *Expectations: Teaching Writing from the Reader's Perspective*, Revised Edition, Amazon. See also www.georgegopen.com

[15] Hall, Glen Richard (1982). Acute triangles in the n-ball, *Journal of Applied Probability*, 19(3), 712-715 DOI: https://doi.org/10.2307/3213533

[16] Hardy, G. H. and Ramanujan, S. (1918). Asymptotic formulae in combinatory analysis, *Proc. London Math. Soc.* 17(2), 75–115.

[17] Hobson, E. W. (1913). *Squaring the Circle*. Oxford University Press. Reprinted by Chelsea Publishing Company, New York.

[18] Holditch, Rev. Hamnet (1858), Geometrical theorem, *The Quarterly Journal of Pure and Applied Mathematics*, 2(38). https://archive.org/details/quarterlyjourna45unkngoog/page/n52/mode/1up

[19] Jones, William (1706). *Synopsis Palmariorum Matheseos: or, a New Introduction to the Mathematics*, pp. 243, 263.

[20] Littlewood, J. E. (1963). *A Mathematician's Miscellany*. New York: Methuen, 85.

[21] Niven, Ivan (1939). The transcendence of π, *Amer. Math. Monthly* 46, 469–471. Reprinted in [5] Appendix A.

[22] Niven, Ivan (July 1947). A simple proof that pi is irrational. *Bulletin of the Amer. Math. Society*, 53 (7): 507. doi:10.1090/S0002-9904-1947-08821-2.

[23] National Geographic Education Video. "The Cartographer's Dilemma". https://www.nationalgeographic.org/video/the-cartographers-dilemma/

[24] Osler, Thomas J. and Waterpeace, Sky (2014). Vieta's product for pi from the Archimedean algorithm. *The Mathematical Gazette* 98(543), 429–431. The Mathematical Association of America.

[25] Pace, Luigi (2011). Probabilistically proving that $\zeta(2) = \pi^2/6$. *Amer. Math. Monthly* 118 (7), 641–643.

[26] Pawan Kumar Biswakarma (2017). Convergence of the ratio of perimeter of a regular polygon to the length of its longest diagonal as the number of sides of polygon approaches to ∞. https://vixra.org/pdf/1709.0144v3.pdf

[27] Pickover, Clifford A. (2009). *The Mαth βook: From Pythagoras to the 57th Dimension, 250 Milestones in the History of Mathematics*. New York: Sterling.

[28] Posamentier, Alfred S. and Lehmann, Ingmar (2004). *Pi: A Biography of the World's Most Mysterious Number*. Prometheus Books. ISBN 978-1-59102-200-8.

[29] Sarkar, Jyotirmoy and Rashid, Mamunur (2022). Mathematical Musings on the External Anatomy of the Novel Corona Virus. *Resonance: The Indian Journal of Science Education*, **27(1)**, 609–622. https://www.ias.ac.in/article/fulltext/reso/027/04/0609-0622

[30] Shirali, Shailesh A. (May 1997). Nilakantha, Euler and pi. *Resonance: The Indian Journal of Science Education*, 2(5), 29–43. https://www.ias.ac.in/article/fulltext/reso/002/05/0029-0043

[31] Wackerly, D. D., Mendenhall, W. and Scheaffer, R. L. (2008). *Mathematical Statistics with Applications*, 7th ed., Brooks/Cole, pp. 277-278.

[32] Wagon, Stan (1985). The evidence: Is π normal? *The Mathematical Intelligencer* 7, 65–67. https://doi.org/10.1007/BF03025811

[33] Wallis, John (2004). *The Arithmetic of Infinitesimals*, translated by Jacquelin A. Stedall. Springer.

[34] Weisstein, Eric W. (no date). Hypersphere, MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/Hypersphere.html

[35] Wells, David (1990). Are these the most beautiful? *Mathematical Intelligencer* 12(3), 37–41. https://www.gwern.net/docs/math/1990-wells.pdf

[36] Wikipedia, the free encyclopedia. Chronology of computatoin of *pi* https://en.wikipedia.org/wiki/Chronology_of_computation_of_π

[37] Wikipedia, the free encyclopedia. (no date). Laplace's method. https://en.wikipedia.org/wiki/Laplace's_method

[38] Wikipedia, the free encyclopedia. (no date). Srinivasa Ramanujan. https://en.wikipedia.org/wiki/Srinivasa_Ramanujan

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